Introduction to Mathematical Quantum Theory

Text of the Exercises

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Teacher: Prof. Chiara Saffirio

Assistent: Dr. Daniele Dimonte – daniele.dimonte@unibas.ch

Exercise 1

Let $k \in \mathbb{Z}$, $d \in \mathbb{N}$, $k+d \neq 0$. Let D be defined as

$$D := \begin{cases} C_c^\infty\left(\mathbb{R}^d\right) & \text{if } k \geqslant 0, \\ C_c^\infty\left(\mathbb{R}^d \setminus \{\mathbf{0}\}\right) & \text{if } k \leqslant -1,\ k+d \neq 0. \end{cases} \tag{1}$$

Prove that for any $\psi \in D$

$$\int_{\mathbb{R}^d} |\mathbf{x}|^k \left|\psi(\mathbf{x})\right|^2 d\mathbf{x} \leqslant \frac{4}{|k+d|^2} \int_{\mathbb{R}^d} |\mathbf{x}|^{k+2} \left|\nabla\psi(\mathbf{x})\right|^2 d\mathbf{x}. \tag{2}$$

Hint: Use the fact that

$$|\mathbf{x}|^k = \frac{1}{k+d} \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(|\mathbf{x}|^k x_j\right) \tag{3}$$

to integrate by part on the left hand side of (2) *and then use the Cauchy-Schwartz inequality.*

Remark: Notice that in particular if $k=-2$ (and $d \neq 2$) this implies that as operators

$$\frac{1}{|\mathbf{x}|^2} \leqslant -\frac{4}{|d-2|}\Delta. \tag{4}$$

A generalisation of this formula is called in the literature the **Hardy inequality**.

Exercise 2

a Let $\mathcal{H} := L^2\left(\mathbb{R}^3\right)$. Define (as in class) the operator H_0 with[1]

$$\mathcal{D}\left(H_0\right) := H^2\left(\mathbb{R}^3\right) \equiv \left\{\psi \in \mathcal{H} \middle|\ |\mathbf{k}|^2 \widehat{\psi}(\mathbf{k}) \in L^2\left(\mathbb{R}^3\right)\right\}, \tag{5}$$

$$H_0\psi = -\Delta\psi = \left(|\mathbf{k}|^2 \widehat{\psi}(\mathbf{k})\right)^\vee, \qquad \forall \psi \in \mathcal{D}\left(H_0\right). \tag{6}$$

Prove that H_0 is closed.

b Let $\mathcal{D}(H) := \mathcal{D}(H_0)$. Define $H := H_0 + \frac{1}{|\mathbf{x}|}$. Prove that H is well-defined and closed. (Assume, if necessary, to know that there exists a positive constant C such that for any $\psi \in H^2\left(\mathbb{R}^3\right)$ it holds $\|\psi\|_{L^\infty} \leqslant C\|\psi\|_{H^2}$).

[1] Recall that we proved in the exercise session that if $\|\psi\|_{H^2} := \left\|\left(1+|\mathbf{k}|^2\right)\widehat{\psi}\right\|_{L^2}$, then $H^2\left(\mathbb{R}^3\right)$ is closed with respect to $\|\cdot\|_{H^2}$.

Hint: Use the fact that $H^2(\mathbb{R}^3) \subseteq L^\infty(\mathbb{R}^3)$ to prove that is well-defined. To prove the closure, use (2) from Exercise 1 to show and subsequently use that $\forall \varepsilon > 0$, $\forall \psi \in \mathcal{D}(H)$

$$\left\| \frac{1}{|\mathbf{x}|}\psi \right\|_{L^2} \leqslant \frac{2}{\varepsilon}\|\psi\|_{L^2} + \varepsilon \|H_0\psi\|_{L^2} \tag{7}$$

to get that

$$\|H_0\psi\|_{L^2} \leqslant \frac{2}{\varepsilon(1-\varepsilon)}\|\psi\|_{L^2} + \frac{1}{1-\varepsilon}\|H\psi\|_{L^2}. \tag{8}$$

c Prove that H is symmetric.

d Prove that H is self-adjoint.

Hint: Use the fact that $\frac{1}{|x|}$ is a self-adjoint operator and apply the Kato-Rellich theorem.

Exercise 3

Let $\mathcal{H}$ an Hilbert space and let $A,\ B \in \mathcal{B}(\mathcal{H})$, $A^* = A$, $B^* = B$

a Suppose[2] $A \geqslant \mathrm{id}$; prove that A is invertible with $A^{-1} \in \mathcal{B}(\mathcal{H})$ and that $0 \leqslant A^{-1} \leqslant \mathrm{id}$.

b Suppose $0 \leqslant A \leqslant B$; prove that for any $\lambda > 0$, $A+\lambda\,\mathrm{id}$ and $B+\lambda\,\mathrm{id}$ are invertible with $(A+\lambda\,\mathrm{id})^{-1},\ (B+\lambda\,\mathrm{id})^{-1} \in \mathcal{B}(\mathcal{H})$ and that we have $(B+\lambda\,\mathrm{id})^{-1} \leqslant (A+\lambda\,\mathrm{id})^{-1}$.

c Suppose $0 \leqslant A \leqslant B$; prove that $\sqrt{A} \leqslant \sqrt{B}$.

Hint: Prove and use the fact that

$$\sqrt{x} = \frac{1}{\pi}\int_0^{+\infty} \frac{1}{\sqrt{\lambda}}\left(1 - \frac{\lambda}{x+\lambda}\right) d\lambda, \qquad \forall x \geqslant 0. \tag{9}$$

Exercise 4

Let $\mathcal{H}$ be an Hilbert space. Let A be a linear self-adjoint operator on $\mathcal{H}$ with $A \geqslant 0$ and $\lambda > 0$. Denote with $\|\cdot\|$ the operator norm and with $\|\cdot\|_{\mathcal{H}}$ the norm induced by the inner product in the Hilbert space $\mathcal{H}$.

a Prove that $\left\|(A+\lambda\,\mathrm{id})^{-1}\right\| \leqslant 1/\lambda$.

b Prove that for all $\psi \in \mathcal{H}$,

$$\|\psi\|_{\mathcal{H}}^2 \geqslant \left\|A(A+\lambda\,\mathrm{id})^{-1}\psi\right\|_{\mathcal{H}}^2 + \lambda^2\left\|(A+\lambda\,\mathrm{id})^{-1}\psi\right\|_{\mathcal{H}}^2. \tag{10}$$

Conclude that $\left\|A(A+\lambda\,\mathrm{id})^{-1}\right\| \leqslant 1$.

[2] Recall that $A \geqslant 0$ if for any $\psi \in \mathcal{D}(A)$, $\langle \psi, A\psi \rangle \geqslant 0$ and that $A \geqslant B$ if $A - B \geqslant 0$.